Analytical solution of the Landau-Lifshitz (LL) equations with conventional damping term

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V. Zayets "Spin rotation after a spin-independent scattering. Spin properties of an electron gas in a solid", *Journal of Magnetism and Magnetic Materials 356 (2014)52–67* *(clich here to download pdf);* (http://arxiv.org/abs/1304.2150) .

The Landau-Lifshitz (LL) equations can be written as

$$\frac{\partial \vec{m}}{\partial t} = -\gamma \vec{m} \times \vec{H} - \lambda \cdot \vec{m} \times \left(\vec{m} \times \vec{H} \right) \quad (1)$$

where $\vec{m}$ is an unit vector directed along the magnetization $|\vec{m}| = 1$

Both the precession and the damping are induced by the same magnetic field H, which is directed along the z- axis.

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ H_z \end{pmatrix}$$

The solution of LL equations (1) :
Temporal evolution of magnetization is described as

$$\begin{aligned} m_x(t) &= \cos(\omega_L t) \cdot \sin(\theta(t)) \\ m_y(t) &= \sin(\omega_L t) \cdot \sin(\theta(t)) \quad (11a) \\ m_z(t) &= \cos(\theta(t)) \end{aligned}$$

where θ is magnetization angle with respect to direction of the magnetic field H and it is calculated as:

$$\theta(t) = 2 \cdot arc\tan\left[e^{-\omega_D \cdot t} \tan\left(\frac{\theta_0}{2} \right) \right] \quad (14a)$$

and

$\omega_L = \gamma H_z$ is the Larmor frequency, $\omega_D = \lambda H_z$ is the damping rate.

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solution

New unknowns are defined as

$$m_{+} = m_x + i \cdot m_y \quad m_{-} = m_x - i \cdot m_y \quad (2)$$

Explicit expressions for vector products are

$$\vec{m} \times \vec{H} = \begin{pmatrix} m_y H_z \\ -m_x H_z \\ 0 \end{pmatrix} \quad \vec{m} \times \left(\vec{m} \times \vec{H}\right) = H_z \begin{pmatrix} m_x m_z \\ m_y m_z \\ -\left(m_x^2 + m_y^2\right) \end{pmatrix} \quad (3)$$

Using (3) and adding/substituting the 1st and 2nd equations of (1) gives

$$\frac{\partial m_{+}}{\partial t} = \frac{\partial \left(m_x + i \cdot m_y\right)}{\partial t} = -\gamma H_z \left(m_y - i \cdot m_x\right) - \lambda H_z m_z \left(m_x + i \cdot m_y\right)$$

$$\frac{\partial m_{-}}{\partial t} = \frac{\partial \left(m_x - i \cdot m_y\right)}{\partial t} = -\gamma H_z \left(m_y + i \cdot m_x\right) - \lambda H_z m_z \left(m_x - i \cdot m_y\right) \quad (4)$$

$$\frac{\partial m_z}{\partial t} = -\lambda H_z \left(m_x^2 + m_y^2\right)$$

where the Larmor frequency ω_L is defined as

$$\omega_L = \gamma H_z \quad (5)$$

and the damping rate ω_D is defined as

$$\omega_D = \lambda H_z \quad (6)$$

The substitution of Eqs. (5,6) into (4) gives

$$\frac{\partial m_{+}}{\partial t} = i\omega_L m_{+} - \omega_D m_z m_{+}$$

$$\frac{\partial m_{-}}{\partial t} = -i\omega_L m_{-} - \omega_D m_z m_{-} \quad (7)$$

$$\frac{\partial m_z}{\partial t} = -\omega_D m_{+} m_{-}$$

The solution of Eq.(7) can be found as

$$\begin{pmatrix} m_{+} \\ m_{-} \end{pmatrix} = m_{xy} \begin{pmatrix} e^{i\omega_L t} \\ e^{-i\omega_L t} \end{pmatrix} \quad (8)$$

After substitution of Eq.(8) into (7), the 1st and 2nd equations of (7) become identical as

$$\frac{\partial m_{xy}}{\partial t} = -\omega_D m_z m_{xy} \quad (9)$$

Combination of Eq.(9) with the 3d equation of (7) gives the system of two differential equations:

$$\frac{\partial m_{xy}}{\partial t} = -\omega_D m_z m_{xy}$$
$$\frac{\partial m_z}{\partial t} = \omega_D m_{xy}^2 \qquad (10)$$

The solution of Eqs. (10) can be found as

$$m_z(t) = \cos(\theta(t))$$
$$m_{xy}(t) = \sin(\theta(t)) \qquad (11)$$

Substitution of Eq. (11) into (10) gives

$$\cos(\theta)\frac{\partial \theta}{\partial t} = -\omega_D \cos(\theta)\sin(\theta)$$
$$-\sin(\theta)\frac{\partial \theta}{\partial t} = \omega_D \sin(\theta)^2 \qquad (12)$$

Two equations of (12) are identical and can be expressed as:

$$\frac{\partial \theta}{\partial t} = -\omega_D \sin(\theta) \quad (13)$$

Integration of Eq.(13) gives

$$\int \frac{d\theta}{\sin(\theta)} = -\omega_D \cdot t + const \quad (14)$$

and integration gives

$$\log\left(\tan\left(\frac{\theta}{2}\right)\right) = -\omega_D \cdot t + const \quad (14)$$

if at time t=0 the θ= θ $_0$

$$\text{const} = \log\left(\tan\left(\frac{\theta_0}{2}\right)\right) \quad (14)$$

or

$$\boxed{\tan\left(\frac{\theta(t)}{2}\right) = e^{-\omega_D \cdot t}\tan\left(\frac{\theta_0}{2}\right) \quad (14)}$$

Summing up

from Eq (2)

$$m_x = \frac{m_+ + m_-}{2} = m_{xy}\cos(\omega_L t) \quad m_y = \frac{m_+ - m_-}{2i} = m_{xy}\sin(\omega_L t) \quad (2a)$$

Combining with Eq. (11), temporal evolution of magnetization is described as

$$m_x(t) = \cos(\omega_L t) \cdot \sin(\theta(t))$$
$$m_y(t) = \sin(\omega_L t) \cdot \sin(\theta(t)) \quad (11a)$$
$$m_z(t) = \cos(\theta(t))$$

where

$$\boxed{\theta(t) = 2 \cdot arc\tan\left[e^{-\omega_D \cdot t} \tan\left(\frac{\theta_0}{2}\right)\right] \quad (14a)}$$